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Estimation of the Maximum Earthquake Magnitude, m_{max}

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Abstract — This paper provides a generic equation for the evaluation of the maximum earthquake magnitude m_{max} for a given seismogenic zone or entire region. The equation is capable of generating solutions in different forms, depending on the assumptions of the statistical distribution model and/or the available information regarding past seismicity. It includes the cases (i) when earthquake magnitudes are distributed according to the doubly-truncated Gutenberg-Richter relation, (ii) when the empirical magnitude distribution deviates moderately from the Gutenberg-Richter relation, and (iii) when no specific type of magnitude distribution is assumed. Both synthetic, Monte-Carlo simulated seismic event catalogues, and actual data from Southern California, are used to demonstrate the procedures given for the evaluation of m_{max}.
|The three estimates of m_{max} for Southern California, obtained by the three procedures mentioned above, are respectively: 8.32 ± 0.43, 8.31 ± 0.42 and 8.34 ± 0.45. All three estimates are nearly identical, although higher than the value 7.99 obtained by FIELD *et al.* (1999). In general, since the third procedure is non-parametric and does not require specification of the functional form of the magnitude distribution, its estimate of the maximum earthquake magnitude m_{max} is considered more reliable than the other two which are based on the Gutenberg-Richter relation.

Key words: Seismic hazard, maximum earthquake magnitude m_{max}.

1. Introduction

This work is aimed at providing a tool that allows for the assessment of the maximum earthquake magnitude, m_{max}.

To avoid confusion about the terminology, in this work the maximum earthquake magnitude, m_{max}, is defined as the upper limit of magnitude for a given seismogenic zone or entire region. Also, synonymous with the upper limit of earthquake magnitude, is the magnitude of the largest possible earthquake. The value of maximum magnitude so defined is the same as that used by many earthquake engineers (EERI COMMITTEE, 1984) and complies with the meaning of this parameter as used by, e.g., HAMILTON (1967), PAGE (1968), COSENTINO *et al.* (1977), the Working Group on California Earthquake Probabilities (WGCEP, 1995), STEIN and HANKS (1998), and FIELD *et al.* (1999). This terminology assumes a sharp cut-off magnitude at a maximum magnitude m_{max}, so that, by definition, no

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earthquakes are possible with a magnitude exceeding $m_{\max}$. Cognizance should be taken of the fact that an alternative, "soft" cut-off maximum earthquake magnitude is also in use (MAIN and BURTON, 1984a; KAGAN, 1991; 2002 a,b). The latter formalism is based on the assumption that seismic moments follow the Gamma distribution. One of the distribution parameters is also called the maximum seismic moment and the corresponding value of earthquake magnitude is called the "soft" maximum magnitude. Beyond the value of this maximum magnitude, the distribution decays much faster than the classical Gutenberg-Richter relation. However, this means that a "soft" cut-off is envisaged since earthquakes with magnitudes larger than such a maximum magnitude are not excluded. Although a model with the "soft" maximum earthquake magnitude has been used by KAGAN (1994, 1997), MAIN (1996), MAIN *et al.* (1999), SORNETTE and SORNETTE (1999) and PISARENKO and SORNETTE (2001), this paper only considers a model having a sharp cut-off of maximum magnitude.

Although a knowledge of the value of the maximum possible earthquake magnitude $m_{\max}$ is required in many engineering applications, it is surprising how little has been done in developing appropriate techniques for an estimation of this parameter. At present there is no generally accepted method for estimating the value of $m_{\max}$. The current methods for its evaluation fall into two main categories: deterministic and probabilistic.

The deterministic procedure most often applied is based on the empirical relationships between magnitude and various tectonic and fault parameters. There are several research efforts devoted to the investigation of such relationships. The relationships are different for different seismic areas and different types of faults (WELLS and COPPERSMITH, 1994; ANDERSON *et al.,* 1996, and the references therein). As an alternative to the above technique, researchers often try to relate the value of $m_{\max}$ to the strain rate or the rate of seismic-moment release (PAPASTAMATIOU, 1980; ANDERSON and LUCO, 1983; WGCEP, 1995; STEIN and HANKS, 1998; FIELD *et al.,* 1999). Such an approach has also been applied in evaluating the maximum possible magnitude of seismic events induced by mining (e.g., MCGARR, 1984). Another procedure for the estimation of $m_{\max}$ was developed by JIN and AKI (1988), where a remarkably linear relationship was established between the logarithm of coda Q_0 and the largest observed magnitude for earthquakes in China. The authors postulate that if the largest earthquake magnitude observed during the last 400 years is the maximum possible magnitude $m_{\max}$, the established relation will give a spatial mapping of $m_{\max}$. A very interesting, alternative procedure for the estimation of $m_{\max}$ was also described by WARD (1997). Ward's computer simulations of the largest earthquake are impressive and convincing. Nevertheless, one must realize that all the quantitative assessments given by WARD (1997) are based on the particular model assumed for the rupture process, on the postulated parameters of the strength of the faults and on the configuration of the faults.

However, in most cases, the uncertainty of the value of the parameter $m_{\max}$ as determined by any deterministic procedure is large, often reaching a value of the order of one unit on the magnitude scale.

In the probabilistic procedures, the value of $m_{\max}$ is estimated purely on the basis of the seismological history of the area, viz. by using seismic event catalogs and an appropriate statistical estimation procedure. The most often used probabilistic procedure for maximum earthquake magnitude was developed in the late sixties, and is based on the extrapolation of the classical, log-linear, frequency-magnitude Gutenberg-Richter relation. Among seismologists and earthquake engineers, the best known is probably the extrapolation procedure as applied recently, e.g., by FROHLICH (1998), and the "probabilistic" extrapolation procedure applied by NUTTLI (1981), in which the frequency-magnitude curve is truncated at the specified value of annual probability of exceedance (e.g., 0.001). Another technique is based on the formalism of the extreme values of random variables. The statistical theory of extreme values was known and well developed in the forties already, and was applied in seismology as early as 1945 (e.g., NORDQUIST, 1945). The appropriate statistical tools required for the estimation of the end-point of distribution functions were developed later (e.g., ROBSON and WHITLOCK, 1964; WOODROOFE, 1972, 1974; WEISS and WOLFOWITZ, 1973; HALL, 1982). However, it was used from the eighties only in estimating maximum earthquake magnitude (DARGAHI-NOUBARY, 1983; KIJKO and SELLEVOLL, 1989, 1992; PISARENKO, 1991; PISARENKO *et al.,* 1996).

The purpose of this paper is to provide a procedure (equation) for the evaluation of $m_{\max}$, which is free from subjective assumptions and which is dependent only on seismic data. The procedure is generic and is capable of generating solutions in different forms, depending on the assumptions about the statistical model and/or the information available about past seismicity. The procedure can be applied in the extreme case when no information about the nature of the earthquake magnitude distribution is available, i.e., the procedure is capable of generating an equation for $m_{\max}$, which is independent of the particular frequency-magnitude distribution assumed. The procedure can also be used when the earthquake catalog is incomplete, i.e., when only a limited number of the largest magnitudes are available.

2. A Generic Equation for the Evaluation of the Maximum Earthquake Magnitude, $m_{\max}$

Presume that in the area of concern, within a specified time interval T, all n of the main earthquakes that occurred with a magnitude greater than or equal to $m_{\min}$ are recorded. Let us assume that the value of the magnitude $m_{\min}$ is known and is denoted as the threshold of completeness. We assume further that the magnitudes are independent, identically distributed, random values with cumulative distribution function (CDF), $F_M(m)$. The unknown parameter $m_{\max}$ is the upper limit of the range of magnitudes and is thus termed the maximum earthquake magnitude, and is to be estimated. Let us assume that all n recorded magnitudes are ordered in ascending order, i.e., $m_1 \leq m_2 \leq \ldots \leq m_n$. We observe that m_n, which is the largest observed magnitude (denoted also as $m_{\max}^{\text{obs}}$), has a CDF

$$F_{M_n}(m) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ [F_M(m)]^n, & \text{for } m_{\min} \leq m \leq m_{\max}. \\ 1, & \text{for } m > m_{\max}. \end{cases} \tag{1}$$

After integrating by parts, the expected value of M_n, $E(M_n)$, is

$$E(M_n) = \int_{m_{\min}}^{m_{\max}} m\, \mathrm{d}F_{M_n}(m) = m_{\max} - \int_{m_{\min}}^{m_{\max}} F_{M_n}(m)\, \mathrm{d}m. \tag{2}$$

Hence

$$m_{\max} = E(M_n) + \int_{m_{\min}}^{m_{\max}} [F_M(m)]^n \mathrm{d}m. \tag{3}$$

Keeping in mind that the value of the largest observed magnitude, $\mathrm{m}_{\max}^{\text{obs}}$, is the best unbiased estimate of $E(M_\mathrm{n})$ (PISARENKO *et al.,* 1996), after replacement of $E(M_\mathrm{n})$ by $\mathrm{m}_{\max}^{\text{obs}}$, equation (3) takes the form

$$m_{\max} = m_{\max}^{\text{obs}} + \int_{m_{\min}}^{m_{\max}} [F_M(m)]^n \mathrm{d}m, \tag{4}$$

in which the desired $m_{\max}$ appears on both sides. However, from this equation an estimated value of $m_{\max}$ (and denoted as $\hat{m}_{\max}$) can be obtained only by iteration. The first approximation of $\hat{m}_{\max}$ can be obtained from equation (4) by replacing the unknown upper limit of integration, $m_{\max}$, by the maximum observed magnitude, $\mathrm{m}_{\max}^{\text{obs}}$. The next approximation is obtained by replacing the upper limit of integration by its previous solution. Some authors simply call the method the *iterative method* and it was found that in most cases the convergence is very fast. An extensive analysis and formal conditions of convergence of the above iterative procedure are discussed, for example, by LEGRAS (1971).

COOKE (1979) was probably the first to obtain this estimator (4) of the upper bound of a random variable[1]. If applied to the assessment of the maximum possible earthquake magnitude, $m_{\max}$, equation (4) states that $m_{\max}$ is equal to the largest magnitude already observed, $\mathrm{m}_{\max}^{\text{obs}}$, increased by an amount $\Delta = \int_{m_{\min}}^{m_{\max}} [F_M(m)]^n \mathrm{d}m$.

[1] It should be noted that in his original paper COOKE (1979) gave an equation in which the upper limit of integration $\mathrm{m}_{\max}^{\text{obs}}$ is rather than $m_{\max}$. Clearly, for large n, when the value of $m_{\max}^{\text{obs}}$ and $m_{\max}$ are close to each other, the two solutions are virtually equivalent.

Equation (4) is, by its nature, very general and has several interesting properties. For example, it is valid for any CDF, $F_M(m)$, and does not require the fulfillment of any additional conditions. It may also be used when the exact number of earthquakes, n, is not known. In this case, the number of earthquakes can be replaced by λT. Such a replacement is equivalent to the assumption that the number of earthquakes occurring in unit time conforms to a Poisson distribution with parameter λ, with T the span of the seismic catalog. It is also important to note that, since the value of the integral Δ is never negative, equation (4) provides a value of $\hat{m}_{\max}$, which is never less than the largest magnitude already observed. Of course, the drawback of the formula is that it requires integration. For some of the magnitude distribution functions the analytical expression for the integral does not exist or, if it does, requires awkward calculations. This is, however, not a real hindrance, since numerical integration with today's high-speed computer platforms is both very fast and accurate. Equation (4) will be called the *generic equation* for the estimation of $m_{\max}$.

In the following section we will demonstrate how equation (4) can be used in the assessment of $m_{\max}$ in the different circumstances that a seismologist or earthquake engineer might face in real life. The three cases to be considered are:

(i) the earthquake magnitudes are distributed according to the doubly-truncated Gutenberg-Richter relation,

(ii) the empirical magnitude distribution deviates moderately from the Gutenberg-Richter relation,

(iii) no specific form of the magnitude distribution is assumed, and only a few of the largest magnitudes are known.

3. *Application of the Generic Equation of m_{max} to three Special Cases*

3.1. CASE I: Use of the Generic Formula when earthquake magnitudes follow the Gutenberg-Richter magnitude distribution. (Formula for $m_{\max}$ for those who accept the Gutenberg-Richter frequency-magnitude distribution unconditionally.)

In this section we will demonstrate how to apply the generic equation (4), when earthquake magnitudes follow the Gutenberg-Richter frequency magnitude distribution.

For the frequency-magnitude Gutenberg-Richter relation, the respective CDF of magnitudes, which are bounded from above by $m_{\max}$, is (Page, 1968)

$$F_M(m) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ \frac{1-\exp[-\beta(m-m_{\min})]}{1-\exp[-\beta(m_{\max}-m_{\min})]}, & \text{for } m_{\min} \leq m \leq m_{\max}, \\ 1, & \text{for } m > m_{\max}, \end{cases} \tag{5}$$

where $\beta = b \ln(10)$, and b is the b parameter of the Gutenberg-Richter relation. Following equation (4), the estimator of $m_{\max}$ requires the calculation of the integral

$$\Delta = \int_{m_{\min}}^{m_{\max}} \left[\frac{1-\exp[-\beta(m-m_{\min})]}{1-\exp[-\beta(m_{\max}^{\mathrm{obs}}-m_{\min})]}\right]^n \mathrm{d}m, \tag{6}$$

an integral which is not simple to evaluate. It can be shown that an approximate, straightforward estimator of $m_{\max}$ can be obtained through the application of Cramér's approximation. According to CRAMÉR (1961), for large n (about 10 and more), the value of $[F_M(m)]^n$ is approximately equal to $\exp\{-n[1-F_M(m)]\}$. Simple calculations show that after replacement of $[F_M(m)]^n$ by its Cramér approximate value, integral (6) takes the form

$$\Delta = \frac{E_1(n_2) - E_1(n_1)}{\beta \exp(-n_2)} + m_{\min} \exp(-n), \tag{7}$$

where $n_1 = n/\{1-\exp[-\beta(m_{\max}-m_{\min})]\}$, $n_2 = n_1 \exp[-\beta(m_{\max}-m_{\min})]$, and $E_1(\cdot)$ denotes an exponential integral function. The function $E_1(\cdot)$ is defined as $E_1(z) = \int_z^\infty \exp(-\zeta)/\zeta \,\mathrm{d}\zeta$, and can be conveniently approximated as $E_1(z) = \frac{z^2+a_1z+a_2}{z(z^2+b_1z+b_2)}$ $\exp(-z)$, where $a_1 = 2.334733$, $a_2 = 0.250621$, $b_1 = 3.330657$, and $b_2 = 1.681534$ (ABRAMOWITZ and STEGUN, 1970). Hence, following equation (4), the estimator of $m_{\max}$ for the Gutenberg-Richter frequency-magnitude distribution is obtained as a solution of the equation

$$m_{\max} = m_{\max}^{\mathrm{obs}} + \frac{E_1(n_2) - E_1(n_1)}{\beta \exp(-n_2)} + m_{\min} \exp(-n). \tag{8}$$

It must be noted that in its current form, equation (8) does not constitute an estimator for $m_{\max}$, since expressions n_1 and n_2, which appear on the right-hand side of the equation, also contain $m_{\max}$. Generally the assessment of $m_{\max}$ is obtained by the iterative solution of equation (8). However, numerical tests based on simulated data show that when $m_{\max} - m_{\min} \leq 2$, and $n \geq 100$, the parameter $m_{\max}$ in n_1 and n_2 can be replaced by $m_{\max}^{\mathrm{obs}}$, thus providing an $m_{\max}$ estimator which can be obtained without iterations.

Equation (8) was introduced in KIJKO and SELLEVOLL (1989). This equation has subsequently been used for the estimation of the maximum possible earthquake magnitude in several seismically active areas such as China (YURUI and TIANZHONG, 1997); Canada (WEICHERT and KIJKO, 1989); Iran (MOTAZEDIAN *et al.*, 1997); India (SHANKER, 1998); Romania (MARZA *et al.*, 1991); Greece (PAPADOPOULOS and KIJKO, 1991); Algeria (HAMDACHE, 1998; HAMDACHE *et al.*, 1998); Italy (SLEJKO and KIJKO, 1991); Spain (GARCIA-FERNANDEZ *et al.*, 1989), Turkey (APTEKIN and ONCEL, 1992; APTEKIN *et al.,* 1992) and the West Indies (ASPINALL *et al.*, 1994). The value of $m_{\max}$ obtained from the solution of equation (8) will be termed the Kijko-Sellevoll estimator of $m_{\max}$, or, in short, K-S.

It should be noted again that the K-S equation for $m_{\max}$ can be used even when the number of seismic events, n, is not known. In such a case, the number of seismic events should be replaced by λT and this replacement is equivalent to the assumption that the number of occurrences conforms with a Poisson distribution which has parameter λ, and T is the timespan of the seismic catalog. Calculation of the variance of the estimated maximum earthquake magnitude, $\mathrm{var}(\hat{\mathrm{m}}_{\max})$, is the same as for Cases II and III, and is shown in Section 3.3.

A significant shortcoming of the K-S equation for $m_{\max}$ estimation comes from the implicit assumptions that (i) seismic activity remains constant in time, (ii) the selected functional form of magnitude distribution properly describes the observations, and (iii) the parameters of the assumed distribution functions are known without error.

3.2. CASE II: Application of the Generic Formula to the Gutenberg-Richter Magnitude Distribution in the case of uncertainty in the b value. (Formula for $m_{\max}$ for those who have limited faith in the Gutenberg-Richter frequency-magnitude distribution.)

In contrast to the assumptions of Case I, that earthquake magnitudes follow the Gutenberg-Richter magnitude distribution, many studies of seismic activity suggest that the seismic process can be composed of temporal trends, cycles, short-term oscillations and pure random fluctuations. A list of well-documented cases of the temporal variation of seismic activity world-wide is given in KIJKO and GRAHAM (1998).

When the variation of seismic activity is a random process, the Bayesian formalism, in which the model parameters are treated as random variables, provides the most efficient tool in accounting for the uncertainties considered above (e.g., DEGROOT, 1970). In this section, a Bayesian-based equation for the assessment of the maximum earthquake magnitude will be derived in which the uncertainty of the Gutenberg-Richter parameter b is taken into account. By allowing for such uncertainty in the b value, it is reasonable to drop the implicit assumptions (i), (ii), and (iii) of Case I.

Following the assumption that the variation of the β value in the Gutenberg-Richter-based CDF (5) may be represented by a Gamma distribution with parameters p and q, the Bayesian (also known as compound or mixed) CDF of magnitudes takes the form (CAMPBELL, 1982):

$$F_M(m) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ C_\beta\left[1 - \left(\frac{p}{p+m-m_{\min}}\right)^q\right], & \text{for } m_{\min} \le m \le m_{\max}, \\ 1, & \text{for } m > m_{\max}, \end{cases} \tag{9}$$

where C_β is a normalizing coefficient. It is not difficult to show that p and q can be expressed in terms of the mean and variance of the β value, where $p = \overline{\beta}/(\sigma_\beta)^2$ and $q = (\overline{\beta}/\sigma_\beta)^2$. The symbol $\overline{\beta}$ denotes the known, mean value of the parameter β, σ_β is the known standard deviation of β describing its uncertainty, and C_β is equal to $\{1 - [p/(p + m_{\max} - m_{\min})]^q\}^{-1}$. Equation (9) is also known (CAMPBELL, 1982) as the Bayesian Exponential-Gamma CDF of earthquake magnitude.

It is important to note that the above way of handling the uncertainty of parameter β is by no means unique. For example, for the same purpose, MORTGAT and SHAH (1979) used a combination of the Bernoulli and the Beta distributions. DONG *et al.* (1984), as well as STAVRAKASIS and TSELENTIS (1987), used a combination of uniform and multinomial distributions. Excellent summaries of alternative ways of handling various uncertainties that are present in the parameters, in the model and in the data, are found in papers by BENDER and PERKINS (1993) and RHOADES *et al.* (1994).

Knowledge of the Bayesian, Gutenberg-Richter distribution (9), makes it possible to construct the Bayesian version of the estimator of $m_{\max}$. Following the generic equation (4), the estimation of $m_{\max}$ requires calculation of the integral

$$\Delta = (C_\beta)^n \int_{m_{\min}}^{m_{\max}} \left[1 - \left(\frac{p}{p + m - m_{\min}}\right)^q\right]^n \mathrm{d}m, \tag{10}$$

which, after application of Cramér's approximation, can be expressed as

$$\Delta = \frac{\delta^{1/q+2} \exp[nr^q/(1 - r^q)}{\beta} [\Gamma(-1/q, \delta \cdot r^q) - \Gamma(-1/q, \delta)], \tag{11}$$

where $r = p/(p + m_{\max} - m_{\min}),\delta = nC_\beta$, and $\Gamma(\cdot, \cdot)$ is the Incomplete Gamma Function. Again, as in the previous case (equation 8), equation (11) does not provide an estimator for $m_{\max}$, since some terms on the right-hand side also contain $m_{\max}$. Thus, the estimator of $m_{\max}$, when the uncertainty of the Gutenberg-Richter parameter b is taken into account, is calculated as an iterative solution of the equation

$$m_{\max} = m_{\max}^{\mathrm{obs}} + \frac{\delta^{1/q+2} \exp[n \cdot r^q/(1 - r^q)}{\beta} [\Gamma(-1/q, \delta \cdot r^q) - \Gamma(-1/q, \delta)] \tag{12}$$

The value of $m_{\max}$ obtained from the solution of equation (12) will be denoted as the Kijko-Sellevoll-Bayes estimator of $m_{\max}$, or, in short, K-S-B. An extensive comparison of performances of K-S and K-S-B estimators is given in KIJKO and GRAHAM (1998).

3.3. Case III: Estimation of $m_{\max}$ when no specific form of the earthquake magnitude distribution is assumed. (Formula for $m_{\max}$ for those who only believe in what they see.)

The procedures derived in the previous sections are parametric and are applicable when the empirical log-frequency-magnitude graph for the seismic series exhibits apparent linearity, starting from a certain $m_{\min}$ value. However, many studies of seismicity show that, in some cases, (i) the empirical distributions of earthquake magnitudes are of bi- or multi-modal character, (ii) the log-frequency-magnitude relation has a strong nonlinear component or (iii) magnitude has the "jump" at the upper end of the empirical distribution (PISARENKO and SORNETTE, 2001), and the presence of "characteristic" events (SCHWARTZ and COPPERSMITH, 1984) is evident. There are, by way of illustration, well-documented cases of such deviations and they include natural seismicity in Alaska (DEVISON and SCHOLZ, 1984), Italy (MOLCHAN *et al.*, 1997), Mexico (SINGH *et al.*, 1983), Japan (WESNOUSKY *et al.*, 1983) and the United States (MAIN and BURTON, 1984b; WEIMER and WYSS, 1997), as well as mine-induced seismicity in the former Czechoslovakia, in Poland and in South Africa (FINNIE, 1994; GIBOWICZ and KIJKO, 1994).

In order to use the generic equation (4) in such cases, the analytical, parametric models of the frequency-magnitude distributions should be replaced by a non-parametric counterpart.

The non-parametric estimation of a probability density function (PDF) is an approach that deals with the direct summation of the kernel functions using sample data. Given the sample data m_i, $i = 1, \ldots, n$, and the kernel function $K(\bullet)$, the kernel estimator $\hat{f}_M(m)$ of an actual, and unknown PDF $f_M(m)$, is

$$\hat{f}_M(m) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{m - m_i}{h}\right), \tag{13}$$

where h is a smoothing factor. The kernel function $K(\bullet)$ is a PDF, symmetric about zero. Its specific choice is not so important for the performance of the method; many unimodal distribution functions ensure similar efficiencies. In this work the Gaussian kernel function,

$$K(\xi) = \frac{1}{\sqrt{2\pi}}\exp\left(-\xi^2/2\right), \tag{14}$$

is used. However, the choice of the smoothing factor h is crucial because it affects the trade-off between random and systematic errors. Several procedures exist for the estimation of this parameter, none of them being distinctly better for all varieties of real data (SILVERMAN, 1986). For purposes of this report the least-squares cross-validation (HALL, 1983; STONE, 1984) was used. The details of the procedure are given by KIJKO *et al.* (2001).

Following the functional form of a selected kernel (14) and the fact that the data come from a finite interval $\langle m_{\min}, m_{\max}\rangle$, the respective estimators of the PDF and CDF of seismic event magnitude are

$$\hat{f}_M(m) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ \dfrac{\left(h\sqrt{2\pi}\right)^{-1}\sum\limits_{i=1}^{n}\exp\left[-0.5\left(\frac{m-m_i}{h}\right)^2\right]}{\sum\limits_{i=1}^{n}\left[\Phi\left(\frac{m_{\max}-m_i}{h}\right)-\Phi\left(\frac{m_{\min}-m_i}{h}\right)\right],} & \text{for } m_{\min} \le m \le m_{\max}, \\ 0, & \text{for } m > m_{\max}, \end{cases} \tag{15}$$

and

$$\hat{F}_M(m) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ \dfrac{\sum_{i=1}^{n}\left[\Phi\left(\frac{m-m_i}{h}\right) - \Phi\left(\frac{m_{\min}-m_i}{h}\right)\right]}{\sum_{i=1}^{n}\left[\Phi\left(\frac{m_{\max}-m_i}{h}\right) - \Phi\left(\frac{m_{\min}-m_i}{h}\right)\right]}, & \text{for } m_{\min} \le m \le m_{\max}, \\ 1, & \text{for } m > m_{\max}. \end{cases} \quad (16)$$

where $\Phi(\xi)$ denotes the standard Gaussian cumulative distribution function.

Despite its flexibility, a model-free technique such as the one above has been used only occasionally in seismology. One of the first uses was in the estimation of the conditional failure rates from successive recurrence times of micro-earthquakes (RICE, 1975). The non-parametric CDF of seismic event occurrence time was also employed by SÓLNES *et al.* (1994). Another application involved the estimation of the spatial distribution of seismic sources (VERE-JONES, 1992; FRANKEL, 1995; CAO *et al.,* 1996; WOO, 1996; BOMMER *et al.,* 1997; JACKSON and KAGAN, 1999; STOCK and SMITH, 2002, and the references there) and the non-parametric estimation of temporal variations of magnitude distributions in mines (LASOCKI and WEGLARCZYK, 1998).

By applying the non-parametric, Gaussian-based assessment of the CDF as given by equation (16), the approximate value of the integral for Δ is (KIJKO *et al.,* 2001)

$$\Delta \cong \int_{m_{\min}}^{m_{\max}} [\hat{F}_M(m)]^n \, \mathrm{d}m = \int_{m_{\min}}^{m_{\max}} \left[\frac{\sum_{i=1}^{n}\left[\Phi\left(\frac{m-m_i}{h}\right) - \Phi\left(\frac{m_{\min}-m_i}{h}\right)\right]}{\sum_{i=1}^{n}\left[\Phi\left(\frac{m_{\max}-m_i}{h}\right) - \Phi\left(\frac{m_{\min}-m_i}{h}\right)\right]}\right]^n \mathrm{d}m. \quad (17)$$

Therefore, the equation for $m_{\max}$ based on the non-parametric Gaussian estimation of the PDF takes the form

$$m_{\max} = m_{\max}^{\text{obs}} + \int_{m_{\min}}^{m_{\max}} \left[\frac{\sum_{i=1}^{n}\left[\Phi\left(\frac{m-m_i}{h}\right) - \Phi\left(\frac{m_{\min}-m_i}{h}\right)\right]}{\sum_{i=1}^{n}\left[\Phi\left(\frac{m_{\max}-m_i}{h}\right) - \Phi\left(\frac{m_{\min}-m_i}{h}\right)\right]}\right]^n \mathrm{d}m. \quad (18)$$

The value of $m_{\max}$ obtained from equation (18) will be denoted as the non-parametric, Gaussian-based estimator or, in short, N-P-G.

The N-P-G estimator of $m_{\max}$ is very useful. Its strongest point is that it does not require specification of the functional form of the magnitude distribution $F_M(m)$. By its nature, therefore, it is capable of dealing with cases of complex empirical distributions, e.g., distributions that are in extreme violation of log-linearity, and/or are multimodal, and/or incorporate "characteristic" earthquakes. The drawback of estimator (18) is that, formally, it requires knowledge of all events with magnitude above a specified level of completeness $m_{\min}$. In practice, though, this can be reduced to knowledge of a few (say 10) of the largest events. Such a reduction is possible because the contribution of the weak events to the estimated value of m_{max} decreases very rapidly as magnitude decreases, and for large n, the few largest observations carry most of the information about its end point, m_{max}. Another drawback of formula (18) is that it requires numerical integration. However, it need not be a real obstacle, since numerical integration with today's PC's is both rapid and accurate.

One should also mention that it is possible to derive another model-free technique for the estimation of m_{max}, which is not based on the formalism of the non-parametric kernel estimation procedure. Such a procedure can be developed by means of order statistics, where the CDF of the magnitude distribution is model-free and is based only on the recorded seismic series.

3.4. Uncertainty in the Determination of m_{max}

Different approaches can be used in the estimation of the accuracy of the above estimators of m_{max}. Essentially, the uncertainty in the determination of m_{max} comes from the random nature of the largest observed magnitude, m_{max}^{obs}. This uncertainty has two components, one, which originates from the random nature of the value of the earthquake magnitude, and the second, which derives from the erroneous determination of its value. Both errors are of an epistemic nature (TORO *et al.*, 1997).

Simple computations show (KIJKO and GRAHAM, 1998) that the approximate variance of the first contribution into the uncertainty of determination of m_{max} is of the order of Δ^2. Assuming that the standard error in the determination of the maximum observed magnitude, m_{max}^{obs}, is known and equal to σ_M, the second contribution to the variance of $\hat{m}_{max}$ is equal to σ_M^2. Therefore, the approximate, total variance of any of the estimators [i.e., (8), (12) and (18)] is given by

$$\mathrm{Var}(\hat{m}_{max}) = \sigma_M^2 + \Delta^2, \tag{19}$$

where the corrections Δ are described by equations (7), (11) and (17) respectively, and the upper limit of integration, m_{max}, is replaced by its estimate, $\hat{m}_{max}$.

Probably the simplest assessment of the exact confidence limits for the estimated value of m_{max} can be obtained by applying the formalism based on the fiducial distribution (KENDALL and STUART, 1969), as applied by PISARENKO (1991). Before deriving the required distribution, we replace the current notation of the CDF of earthquake magnitude, $F_M(m)$, by F_M (m; m_{max}), which explicitly shows that the maximum magnitude, m_{max}, is one of the parameters of the magnitude distribution. Following the procedure developed by PISARENKO (1991), a $100(1-\alpha)$ % upper confidence limit on the estimated maximum earthquake magnitude $\hat{m}_{max}$ can be written as

$$\Pr[m_{max} < F_M^{-1}(m_{max}^{obs}; \alpha^{1/n})] = 1 - \alpha, \tag{20}$$

where $F_M^{-1}(m; \bullet)$ denotes an inverse of the cumulative distribution function $F_M(m; m_{\max})$. Knowledge of the above equation makes it possible to construct the distribution of $m_{\max}$

$$\Pr[m_{\max} < z] = 1 - [F_M(m_{\max}^{\text{obs}}; z)]^n, \tag{21}$$

known as the fiducial distribution. Equation (21) describes the confidence limits for any actual value of $m_{\max}$ which can be used when the parameters of the distribution $F_M(m)$ and the maximum observed magnitude $m_{\max}^{\text{obs}}$ are known. Again, after accepting the assumption that the number of seismic events, n, obeys the Poisson distribution with parameter λ, after the replacement $n = \lambda T$, where T denotes the time span of the catalogue, one obtains a distribution of $m_{\max}$ independent of the number of observations, n. The simplicity of equation (21) makes it very attractive. Also it is interesting to note, that when $z \to +\infty$, the probability (21) tends to some value less than unity. This means that with probability $\alpha_0 = 1 - \Pr[m_{\max} = +\infty] = [F_M(m_{\max}^{\text{obs}}; +\infty)]^n$, the current information (seismic event catalogue, applied model of frequency-magnitude distribution and its parameters), is inadequate and/or insufficient for the reliable assessment of $m_{\max}$. One can find more information on this interesting subject in the paper by PISARENKO (1991).

4. Tests of the Procedures using Monte-Carlo Simulations

Simulated catalogues were used to determine the accuracy with which $m_{\max}$ was estimated by the procedures given in the previous sections. The tests were designed to answer three basic questions: (1) How does the accuracy of the estimated maximum earthquake magnitude depend on the number of events in the catalogue? More precisely, what is the minimum number of events required to estimate $m_{\max}$ with sufficient accuracy (say to 0.1 unit of magnitude)? (2) How do the K-S, K-S-B and N-P-G solutions of $m_{\max}$ behave in the presence of "reasonable" differences between the assumptions used in their derivation and the true model of the frequency-magnitude distribution? (3) If it is true that only the largest events provide information on $m_{\max}$, how many such events are required to assess $m_{\max}$ with sufficient accuracy?

4.1. The Minimum Number of Events Required to Assess $m_{\max}$

To answer the first question 1000 simulated catalogues were generated, with the b value equal to exactly 1 ($\beta \cong 2.30$), with the "true" $m_{\max} = 8.0$, and with $m_{\min} = 7.0$, 6.0, and 5.0, respectively. The simulations were performed for different numbers of earthquakes, ranging from 50 to 500. All of the generated magnitudes were rounded off to the first decimal place.

The results of the estimation of $m_{\max}$ by the K-S procedure for the respective 3 levels of completeness ($m_{\min} = 7.0$, 6.0, and 5.0) are given in Figure 1. All

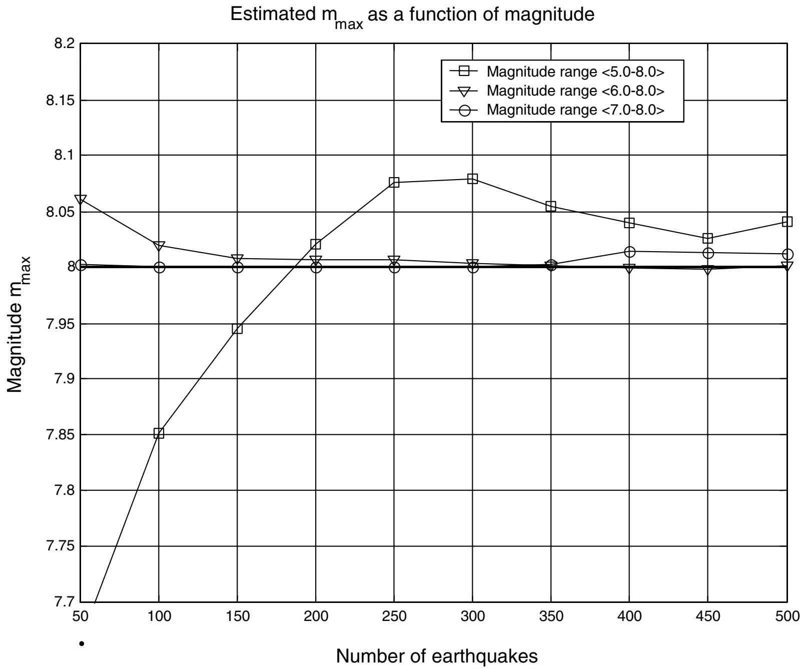

Figure 1
Performance of the K-S estimator for a magnitude range from 1 to 3. Each of the maximum magnitude $m_{\max}$ estimates are based on 1000 synthetic catalogues with magnitudes distributed according to the doubly-truncated Gutenberg-Richter relation with a b value equal to 1. When the magnitude range $\langle m_{\min}, m_{\max}\rangle$ does not exceed 2 units of magnitude (lines with triangular and circle markers), then 50 events on average are sufficient to assess the value of $m_{\max}$. If the range is close to 3 units of magnitude (line with square markers), an accurate assessment of $m_{\max}$ requires at least 150 events.

calculations for $m_{\max}$ were performed using a β value that was obtained empirically, according to the maximum likelihood procedure developed by PAGE (1968). The results of the $m_{\max}$ evaluation in the cases when the range $\langle m_{\min}, m_{\max}\rangle$ is equal to one, two and three units of magnitude are shown by the circlular, triangular and square markers, respectively. Figure 1 indicates that 50 events, on average, are sufficient for the assessment of the value of $m_{\max}$, i.e. when the difference between $m_{\max}$ and the level of completeness $m_{\min}$ does not exceed two units of magnitude. If the magnitude range is equal to three, the formula works well for about 150 events or more. This numerical experiment is important because it provides a lower limit on the number of seismic events required for a reliable assessment of $m_{\max}$. Conclusions drawn from these numerical experiments are correct not only for the values of $m_{\min}$ and $m_{\max}$ actually used, but for any values of $m_{\max}$ and $m_{\min}$, provided that the difference between them is the same as in the experiment, and that the b value of Gutenberg-Richter relation is close to 1.

4.2. The Behavior of the K-S, K-S-B and N-P-G Estimators of m_{max} *when Data are Generated by Different Frequency-magnitude Distributions*

Simulated magnitudes were generated using three different models for their frequency-magnitude distribution namely:

– Model I: classical, doubly-truncated Gutenberg-Richter (equation 5),
– Model II: Bayesian Gutenberg-Richter (equation 9), and
– Model III: a mixture of the Gutenberg-Richter and characteristic earthquake distributions.

The parameters of the three models are given in Table 1. The results of the m_{max} assessments for Models I, II and III are shown in Figures 2, 3 and 4, respectively. For each model there are three estimates for m_{max}: K-S, K-S-B and N-P-G. Again, all the estimates are obtained from averaging the values of m_{max} calculated from 1000 catalogues of which the range in the number of events in each catalogue is between 50–500.

Estimation of m_{max} *based on Model I. (Synthetic data are generated according to the classical Gutenberg-Richter relation)*. The mean values of the non-parametric (N-P-G) and the parametric (K-S and K-S-B) estimates of m_{max} for model I, with $m_{min} = 6.0$, $m_{max} = 8.0$ and $b = 1.0$ are presented in Figure 2. When the number of events is less than about 100, all three estimators are slightly biased. The bias of the non-parametric estimate, N-P-G, is negative, while the bias of the parametric estimators, K-S and K-S-B, is positive. In both cases the bias is low—it does not exceed 0.1 unit of magnitude. As one might expect, both parametric procedures provide similar results. The bias decreases as the number of events increases. In absolute terms, the non-parametric estimate of m_{max} is not significantly worse than its parametric counterpart. In the above experiment a moderate difference of 2 units of magnitude between m_{max} and m_{min} was chosen. If the difference between m_{max} and m_{min} is smaller (Fig. 1, line with circular markers), estimation of m_{max} with the same accuracy requires significantly fewer events.

Table 1

Models of the magnitude distribution that were tested and their respective parameters used to generate the synthetic catalogs

Model	Parameters
Gutenberg-Richter (equation 5)	$b = 1.0$ ($\beta = 2.30$) $m_{min} = 6.0$, $m_{max} = 8.0$
Bayesian-Gutenberg-Richter (equation 9)	$b = 1.0$ ($\beta = 2.30$), $\sigma_b = 0.25$ $m_{min} = 6.0$, $m_{max} = 8.0$
0.95 Gutenberg-Richter + 0.05 Uniform	Parameters of Gutenberg-Richter distribution: $b = 1.0$ ($\beta = 2.30$), $m_{min} = 5.0$, $m_{max} = 7.0$ Parameters of uniform distribution: $m_{min} = 7.0$, $m_{max} = 8.0$

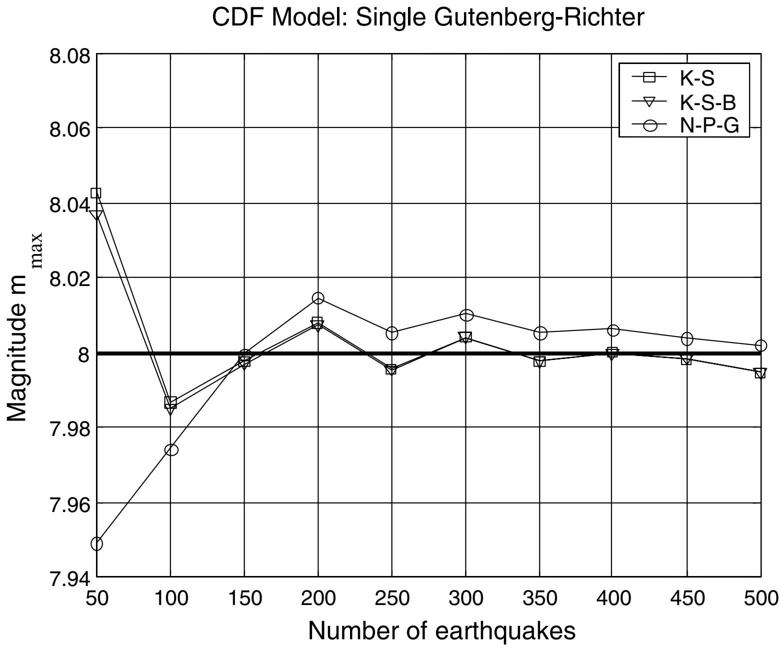

Figure 2

Performance of the three derived estimators for model I (viz. the classic frequency-magnitude Gutenberg-Richter relation). Each estimate of m_{max} is based on 1000 synthetic catalogues, in which the "true" value of $m_{max} = 8.0$, $m_{min} = 6.0$, and $b = 1$. Both parametric estimators (viz. K-S and K-S-B) provide similar results. When the model of magnitude distribution assumed is the same as that of the distribution of data, the non-parametric estimate of m_{max} is not significantly worse than its parametric counterparts, K-S, and K-S-B.

Estimation of m_{max} *based on Model II. (Synthetic data are generated according to the Gutenberg-Richter relation with fluctuating b value.)* The performance of the three estimators for model II, describing the presence of uncertainties in the b value, is shown in Figure 3. This comparison was based on 1000 synthetic catalogues in which the "true" value of m_{max} was 8.0, m_{min} was 6.0, and the b value was subjected to a random, normally distributed error with mean equal to zero and the standard deviation equal to 25% of the b value. The K-S estimator (which, by its nature, ignores the uncertainty in the b value) is shown in Figure 3 where the value of m_{max} is significantly overestimated. The superiority of the K-S-B estimator, which explicitly takes into account the uncertainty in the b value over the K-S procedure, is clearly seen. Again, the non-parametric estimate of m_{max}, which is slightly biased in the case of a small number of events, is essentially the same as its parametric counterpart, K-S-B. As a result: if one were to select the wrong magnitude distribution model for a particular dataset, the parametric K-S procedure can largely overestimate the value of m_{max}.

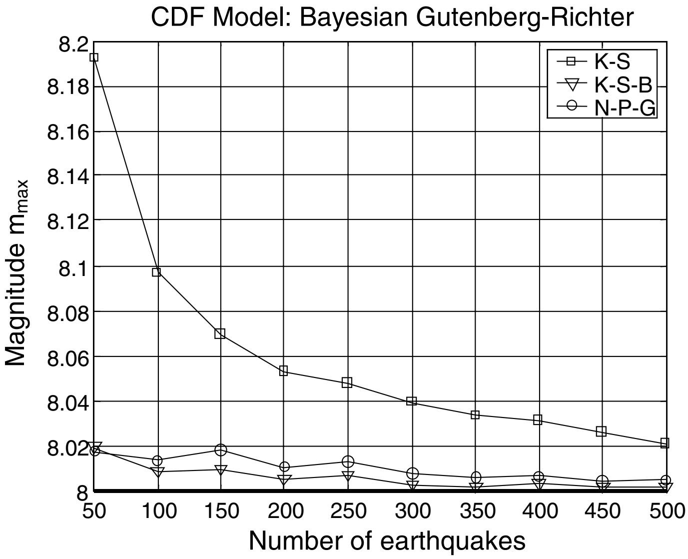

Figure 3
Performance of the three derived estimators for model II, describing the presence of uncertainties in the Gutenberg-Richter parameter, b. Each estimate of m_{max} is based on 1000 synthetic catalogues, in which the "true" value of $m_{max} = 8.0$, $m_{min} = 6.0$, the mean value of $b = 1$, and the b value was subjected to a random, normally distributed error with mean equal to zero and standard deviation equal to 0.25. The K-S estimator ignores the uncertainty in the b value and significantly overestimates m_{max}. The superiority of the K-S-B estimator, which accounts for uncertainty in the b value over the K-S procedure, is clearly seen. The non-parametric estimate of m_{max} is only slightly biased for a small number of events, and is essentially the same as K-S-B.

Estimation of m_{max} *based on Model III. (Synthetic data are generated according to a mixture of the classical Gutenberg-Richter relation and characteristic events.)* The above conclusions are supported by results of the subsequent experiment shown in Figure 4 where the results of the estimation of m_{max} for model III (viz. the Gutenberg-Richter + Characteristic Earthquakes) is presented. The K-S estimation, which is designed to assess the value of m_{max} for the pure Gutenberg-Richter distribution, and which makes no provision for deviation from this model, significantly overestimates the value of m_{max}. The same overestimation is yielded (but not to such an extent) by the second parametric estimator, viz. K-S-B. The positive bias becomes insignificant when the number of events approaches 200. The non-parametric procedure overestimates the true value of m_{max}, but only slightly. Again, the positive bias is insignificant when the number of events exceeds about 200.

4.3. The Number of Largest Events Required to Assess m_{max} *with Sufficient Accuracy*

The final experiment was designed to verify the opinion, often stated (e.g., DARGAHI-NOUBARY, 1983), that in order to assess the value of m_{max}, it is not

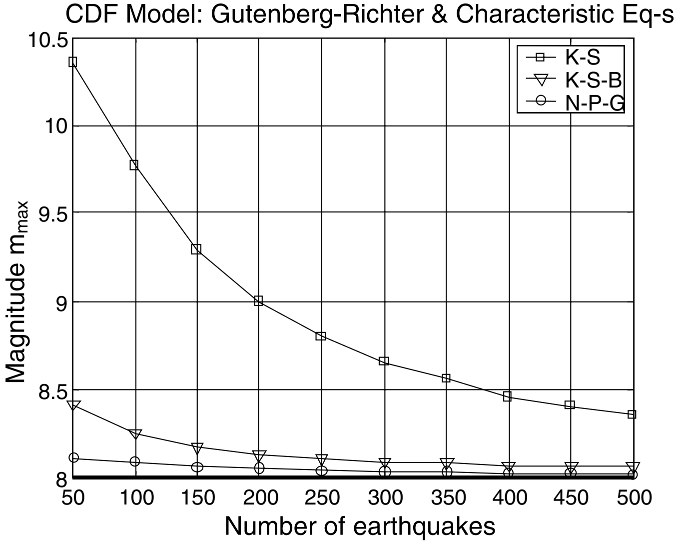

Figure 4
Performance of the three derived estimators for model III, taking into account the presence of characteristic earthquakes. Each estimate of $m_{\max}$ is based on 1000 synthetic catalogues, in which the "true" value of $m_{\max} = 8.0$, $m_{\min} = 6.0$ and the b value of the Gutenberg-Richter relation is 1. Both parametric estimators (K-S and K-S-B) significantly overestimate the value of $m_{\max}$. The test shows that when the model of magnitude distribution selected is wrong, the parametric approach can result in unacceptably large errors. At the same time the non-parametric procedure overestimates the value of $m_{\max}$ only slightly.

necessary to know a large number of events. It is considerably more important to know the "proper" events, viz. the strongest ones, since the largest events bring the most information concerning the upper end of the magnitude distribution function. The results of the estimation of $m_{\max}$ by the non-parametric procedure N-P-G that was applied only to the 5, 10 and 25 largest events are shown in Figure 5. Again, as in all previous experiments, 1000 synthetic catalogues were generated for a range of magnitudes equal to 2 ($m_{\max} = 8.0$, $m_{\max} = 6.0$), and b value equal to 1.0. As one might expect, the largest negative bias in the estimation of $m_{\max}$ is produced by that curve for which only the 5 largest events were used. The best estimate however is obtained when all the events are used. When the number of events in the catalogue exceeds ca. 100, all the curves (viz. those based on the 5, 10 and 25 largest events) provide a value of $m_{\max}$ with an error of less than 0.1.

5. Example of Determination of $m_{\max}$ for Southern California

All information pertaining to the seismicity of Southern California during the last 150 years was taken from Appendix A of a paper by Field *et al.* (1999). In order to

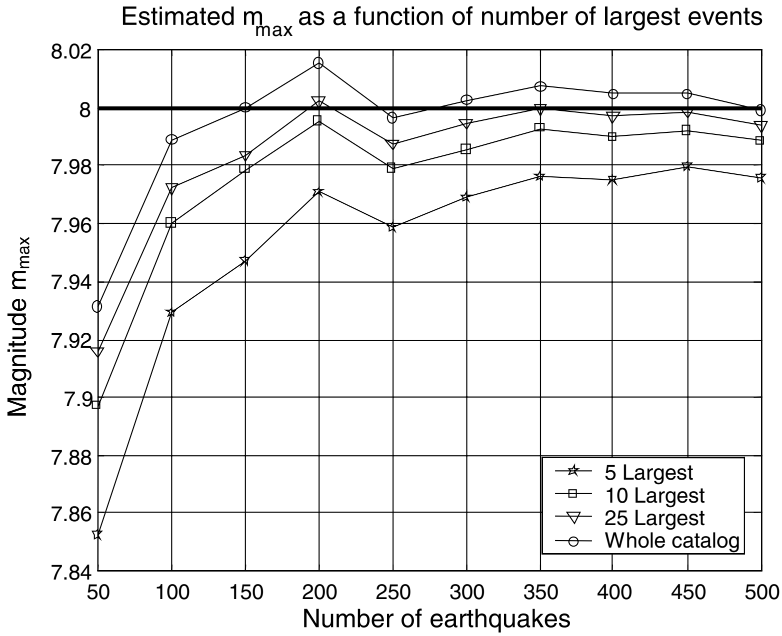

Figure 5
Performance of the non-parametric, N-P-G, estimator when applied to 5, 10 and 25 largest events. Each estimate of $m_{\max}$ is based on 1000 synthetic catalogues, in which the "true" value of $m_{\max} = 8.0$, $m_{\min} = 6.0$ and the b value of the Gutenberg-Richter relation is 1.

be consistent with the assumption of the independence of seismic events (required by estimators K-S and K-S-B), all aftershocks were removed. This reduced catalog also has different levels of completeness for various time intervals. Application of the maximum likelihood procedure to this catalog (KIJKO and SELLEVOLL, 1992), yields the values: $\hat{\lambda}(m_{\min} = 5.0) = 2.14 \pm 0.17, \hat{b} = 0.79 \pm 0.06$, $\hat{m}_{\max}^{K-S} = 8.32 \pm 0.43$ and $\hat{m}_{\max}^{K-S-B} = 8.31 \pm 0.43$, where $\hat{m}_{\max}^{K-S}$ and $\hat{m}_{\max}^{K-S-B}$ denote, respectively, the K-S estimator (8) and the K-S-B estimator (12).ü[2] Application of the remaining procedure to find estimates of $m_{\max}$ yields: $\hat{m}_{\max}^{N-P-G} = 8.34 \pm 0.45$, where $\hat{m}_{\max}^{N-P-G}$ is the non-parametric, Gaussian-based estimator (18). The observed, cumulative

[2] It is noteworthy that soon after its development (1987–1988), the maximum likelihood procedure as applied above was compared with a similar technique developed by WEICHERT (1980) A summary of a comparison between the two techniques is given WEICHERT and KIJIKO (1989). Extensive tests based on synthetic catalogs show that for a given value of $m_{\max}$, both procedures are equivalent and produce the same results. The main difference between the two techniques lies in the fact that Weichert's procedure requires *a priori* knowledge of the maximum magnitude, while the Kijko-Sellevoll approach provides its own estimation. In addition, the latter procedure permits the combination of the largest (earlier) earthqukes with (later) complete data and explicityly takes into account the uncertainty in determination of magnitude.

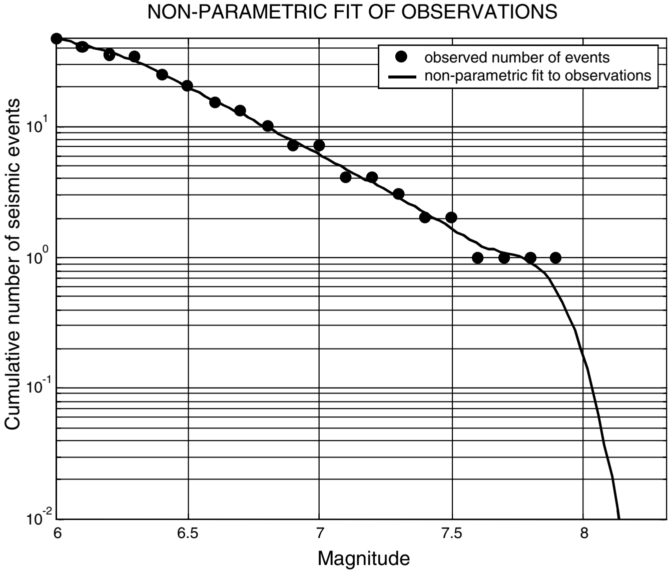

Figure 6
Plot of observed cumulative number of earthquakes (after FIELD *et al.*, 1999) and the non-parametric fit (based on CDF (16)) for the data from Southern California. The estimated value of $m_{\max}$ from the fit is equal to 8.34.

number of earthquakes and its non-parametric fit for the data from Southern California are shown in Figure 6. The value of the smoothing factor h was estimated as 0.12. All estimated values of $m_{\max}$ together with their standard errors are shown in Table 2. Standard errors of estimated values of $m_{\max}$ were calculated according to formula (19) for the standard error of maximum observed magnitude $\sigma_M$equal to 0.25. This value is chosen arbitrarily.

All three estimated parameters differ from the corresponding values obtained by FIELD *et al.* (1999), in which the least-squares fit of all data gives $\hat{\lambda}(m_{\min} = 5.0) = 3.33$, $\hat{b} = 0.92$, and $\hat{m}_{\max} = 7.99$. Clearly, the differences follow

Table 2

The estimated values of m_{max} and their standard errors. The values in the last column give the probabilities that the current data and the applied model are sufficient to assess the value of m_{max}, as obtained by the three procedures developed in this paper for Southern California. The last row shows the value of m_{max} = 7.99 as obtained by FIELD et al. (1999)

Procedure	$\hat{m}_{\max} \pm SD$	$1 - \alpha_0$
K-S	8.32 ± 0.43	0.76
K-S-B	8.31 ± 0.42	0.86
N-P-G	8.34 ± 0.45	0.61
(based on non-parametric estimation of PDF)		
FIELD *et al.* (1999)	7.99	

from the different assumptions, the different models, the application of different estimation procedures and the use of different data. Our assessments are based on the Gutenberg-Richter relation only (for the K-S and K-S-B estimators), while the FIELD *et al.* (1999) model contains an additional component—the occurrence of characteristic earthquakes. Furthermore, the FIELD *et al.* (1999) model has its whole procedure constrained by the principle of conservation of seismic moment. Our estimates are based on the maximum likelihood principle, while the Field *et al.* (1999) results originate from the least-squares fit. The FIELD *et al.* (1999) results are based on all available data (main events and aftershocks), while our estimates are based only on main earthquakes.

The fiducial distribution of $m_{\max}$ calculated according to the K-S, K-S-B and N-P-G procedures is shown in Figures 7 and 8. The respective probabilities $1 - \alpha_0$ for the three applied techniques are shown in Table 2. All 3 values of $1 - \alpha_0$ are relatively low which indicates that all the estimated values of $m_{\max}$ are unreliable. According to PISARENKO (1991), the assessment of $m_{\max}$ is reliable and stable when the value of $1 - \alpha_0$ is equal to 0.90 and higher. The low value of $1 - \alpha_0$ can be attributed to short periods of observations, which in the case of Southern California is equal to ca. 150 years. Again, following PISARENKO (1991), in general, the span of the seismic catalogue is considered to be sufficient if at least 2–3 earthquakes took place with magnitudes close to $m_{\max}$ (with a difference of the order of 0.3–0.4). This condition is not fulfilled by the current catalogue for Southern California, since the three strongest earthquakes that took place during the last 150 years are 7.9, 7.5 and 7.3 magnitude units and the estimated maximum possible magnitude for the area, $\hat{m}_{\max}$, is close to 8.3.

For that reason, it is rather surprising that the solutions of the three equations discussed give such similar values for the maximum possible earthquake magnitude for Southern California. Of course, it might be coincidental. In general, since the N-P-G procedure is, by its nature, non-parametric, and does not require specification of the functional form of the magnitude distribution, this procedure is considered more reliable than the model-based estimators K-S and K-S-B.

6. Discussion, Remarks and Conclusions

This paper is aimed at the determination of the maximum possible earthquake magnitude, $m_{\max}$, for a given seismogenic zone or the entire region. A generic equation for the evaluation of $m_{\max}$ was developed. The equation is very flexible and is capable of generating solutions in different forms, depending on the assumptions relative to the model and/or regarding the available information on past seismicity. Three special cases of the generic equation were discussed, namely:

— when earthquake magnitudes are distributed according to the doubly-truncated Gutenberg-Richter relation,

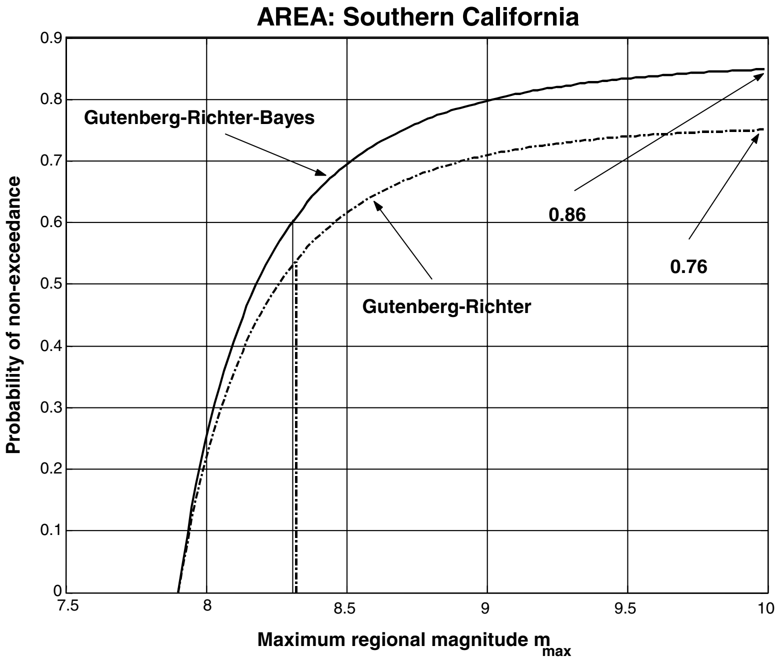

Figure 7
Fiducial distribution function for $m_{\max}$, Southern California, when the Gutenberg-Richter and Gutenberg-Richter-Bayes model of earthquake magnitude distributions are assumed. The vertical lines show the median values of $m_{\max}$. The respective probabilities, $1 - \alpha_0$, in as much as the current data and the applied model are sufficient to assess the value of $m_{\max}$, are equal to 0.76 (Gutenberg-Richter model) and 0.86 (Gutenberg-Richter-Bayesian) model. In both cases the median values of $m_{\max}$ are close to each other, and are equal to 8.26 and 8.31, respectively.

— when the empirical magnitude distribution deviates moderately from the Gutenberg-Richter relation, and
— when no specific form of magnitude distribution is assumed.

The first two solutions of the generic equation (4) provide estimators of $m_{\max}$ that are parametric, which have the same parameters as used in the description of the CDF of magnitude distribution. Since the third solution of the generic equation does not require specification of the functional form of the magnitude distribution, the estimator of $m_{\max}$ obtained is non-parametric.

Tests performed on simulated seismic event catalogues are intended to model the typical scenarios presented in the assessment of seismic hazard. Two types of scenarios are simulated: when the assumed model of magnitude distribution is the same as the empirical distribution of data, and when the assumed model of magnitude distribution is wrong.

It is shown that when earthquake magnitudes rigorously follow the model of the magnitude distribution assumed (the Gutenberg-Richter relation with the b value close to 1 is considered), and the range of earthquake magnitudes $\langle m_{\min}, m_{\max}\rangle$ does

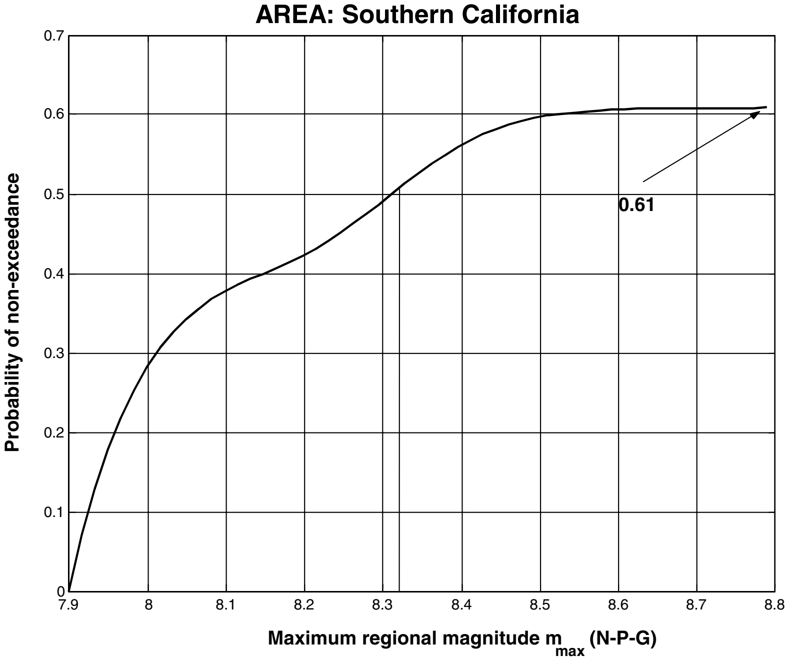

Figure 8
Fiducial distribution function for $m_{\max}$, Southern California, when the empirical distribution of magnitude is estimated according to the N-P-G procedure. The probability, $1 - \alpha_0$, since the current data and applied model are sufficient to assess the value of $m_{\max,}$ is equal to 0.61. The median value of $m_{\max}$ is equal 8.32.

not exceed 2 units, then, on average, 50 events are enough to assess the value of $m_{\max}$. If the range of magnitudes is near 3, then an accurate assessment of $m_{\max}$ requires at least 150 events.

It is demonstrated that when the model of the assumed magnitude distribution is the same as that of the data distribution, then the non-parametric estimates of $m_{\max}$ are not significantly worse than the estimates provided by the parametric approach. On the contrary, when the model of the selected magnitude distribution is wrong, the parametric approach can result in an unacceptably erroneous estimation of $m_{\max}$.

Further, the common opinion that the value of $m_{\max}$ can be estimated on the basis of knowing only the few strongest events, was tested. It is shown that for a typical scenario (Gutenberg-Richter b value equal to 1.0 and the range of magnitude not exceeding two units), it is enough to know only the five largest events from a catalogue of 100 events to assess the value of $m_{\max}$ with an error less than 0.1.

The three estimators derived are applied in assessing the value of the maximum earthquake magnitude for Southern California. The three estimates of $m_{\max}$, using the respective estimators (8), (12) and (18), are: 8.32 $\pm$ 0.43, 8.31 $\pm$ 0.42 and 8.34 $\pm$ 0.45. These estimates overlap when their standard deviations are taken into account. Once more it should be noted that estimated standard errors of maximum possible magnitudes, $\hat{m}_{\max}$, depend on the chosen standard errors of maximum observed magnitudes, σ_M (see equation 19). In this study a rather high but arbitrary value of $\sigma_M = 0.25$ was chosen, making the standard error in the estimate also conservative.

The values of $\hat{m}_{\max}$ for Southern California obtained from the two parametric procedures (K-S and K-S-B procedure), differ slightly from the value obtained from the non-parametric procedure, N-P-G. These differences can be attributed to the fact that the first two estimators are based on the Gutenberg-Richter model of the frequency-magnitude relation, which might not be correct for Southern California.

In general, since the N-P-G procedure is non-parametric and does not require specification of the functional form of the magnitude distribution, its estimate of the maximum possible magnitude $m_{\max}$, is more reliable than the model-based estimators K-S and K-S-B.

It should be noted that the applied formalism provides not only a confidence limit for the estimated maximum possible earthquake magnitude, $m_{\max}$, but also gives a simple indicator as to how reliable the estimated maximum magnitude is.

Although the proposed procedure for assessment of the maximum earthquake magnitude $m_{\max}$ is very general and tractable, it has significant shortcomings. In its present form the procedure does not allow for the introduction of any additional constraints, e.g., the conservation of seismic moment, the slip rate or the strain rate. In a follow-up paper a similar procedure will be developed that allows for additional constraints.

The computer program used for the maximum likelihood estimation of the mean value of the seismic activity rate, λ, the Gutenberg-Richter parameter, b, the K-S and the K-S-B estimators of $m_{\max}$, using incomplete and uncertain data files, is available on request from the author at e-mail address: kijko@geoscience.org.za. Alternatively, the program can be downloaded from the Council for Geoscience website at http://www.geoscience.org.za/seismo. A detailed description of the estimation procedure and the technique in which magnitude uncertainties and incompleteness of the catalogue are incorporated, can be found in KIJKO and SELLEVOLL (1992).

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References

ABRAMOWITZ, M., and STEGUN, I.A., *Handbook of Mathematical Functions* (9th ed.) (Dover Publ., New York 1970).

ANDERSON, J.G., and LUCO, J.E. (1983), *Consequences of Slip rate Constrains on Earthquake Occurrence Relation*, Bull. Seismol. Soc. Am. *73*, 471–496.

ANDERSON, J.G., WESNOUSKY, S.G., and STIRLING, M.W. (1996), *Earthquake Size as a Function of Slip Rate*, Bull. Seismol. Soc. Am. *86*, 683–690.

APTEKIN, O., and ONCEL, A.O. (1992), *Effects of Magnitude Errors in Seismic Risk Estimates and the Seismic Risk in Erzincan and Vicinity*, Jeofizik *6*, 35–53. (in Turkish with English abstract).

APTEKIN, O., ONCEL, A.O., and YORUK, A. (1992), *Estimation of Seismic Risk for the North Anatolian Fault Zone by Maximum Likelihood Method*, Jeofizik *6*, 35–53. (in Turkish with English abstract).

ASPINALL, W.P., SHEPHERD, J.B., WOO, WIGHTMAN, G.A., ROWLEY, K.R. LYNCH, L.L., and AMBEH, W.B. (1994), *Seismic Ground motion Hazard Assessment at a Site near a Segment Subduction Zone:The Roseau Dam, Saint Lucia, West Indies*, Earthquake Spectra *10*, 259–292.

BENDER, B.K., and PERKINS, D.M. (1993), Treatment of Parameter Uncertainty and Variability for Single Seismic Hazard Map, Earthquake Spectra *9*, 165–195.

BOMMER, J., MCQUEEN, C., SALAZAR, W., SCOTT, S., and WOO, G., *Spatial distribution of seismic hazard in El Salvador. In Proc. Seminario sobre Evaluación y Mitigación del Riesgo Sísmico en el Area Centroamericana* (Universidad Centroamericana, San Salvador 1997).

CAMPBELL, K.W. (1982), *Bayesian Analysis of Extreme Earthquake Occurrences. Part I. Probabilistic Hazard Model*, Bull. Seismol. Soc. Am. *72*, 1689–1705.

CAO, T., PETERSEN, M.D. and REICHLE, M.S. (1996), *Seismic Hazard Estimate from Background Seismicity in Southern California*, Bull. Seismol. Soc. Am. *86*, 5, 1372–1381

COOKE, P. (1979), *Statistical Inference for Bounds of Random Variables*, Biometrika *66*, 367–374.

COSENTINO, P., FICARA, V., and LUZIO, D. (1977), *Truncated Exponential Frequency-magnitude Relationship in the Earthquake Statistics*, Bull. Seismol. Am. *67*, 1615–1623.

CRAMÉR, H., *Mathematical Methods of Statistics* (Princeton University Press, Princeton 1961).

DARGAHI-NOUBARY, G.R. (1983), *A Procedure for Estimation of the Upper Bound for Earthquake Magnitudes*, Phys. Earth Planet. Interiors *33*, 91–93.

DEGROOT, M.H., *Optimal Statistical Decisions* (McGraw-Hill, New York 1970).

DEVISON, F.C. and SCHOLZ, C.H. (1984), *The Test of the Characteristic Earthquake Model for the Aleutian Arc* (abstract), EOS *65*, 242.

D*ong*, W.M., S*hah*, H.C., and B*ao*, A.B. (1984), *Utilization of Geophysical Information in Bayesian Seismic Hazard Model*, Soil Dynamics and Earthquake Engineering, *3*, 103–111.

EERI COMMITTEE ON SEISMIC RISK (H.C. Shah, Chairman) (1984), *Glossary of Terms for Probabilistic Seismic Risk and Hazard Analysis*, Earthquake Spectra *1*, 33–36.

FIELD, D.H., JACKSON, D.D., and DOLAN, J.F. (1999), *A Mutually Consistent Seismic-hazard Source Model for Southern California*, Bull. Seismol. Soc. Am. *89*, 559–578.

FINNIE, G. (1994), *A Stationary Model for Time-dependent Seismic Hazard in Mines*, Acta Geophys. Pol. *42*, 111–118.

FRANKEL, A. (1995), *Mapping Seismic Hazard in the Central and Eastern United States,* Seismol. Res. Lett. *66*, 8–21.

FROHLICH, C. (1998), *Does Maximum Earthquake Size Depend on Focal Depth*? Bull. Seismol. Soc. Am. *88*, 329–336.

GARCIA-FERNANDEZ, M., JIMENEZ, M., and KIJKO, A. (1989), *Seismic Hazard Parameters Estimation in Spain from Historical and Instrumental Catalogues*, Tectonophysics *167*, 245–251.

GIBOWICZ, S.J. and KIJKO, A., *An Introduction to Mining Seismology* (Academic Press, San Diego 1994).

HALL, P. (1982), *On Estimating the Endpoint of a Distribution*, Ann. Statist. *10*, 556–568.

HAMDACHE, M. (1998), *Seismic Hazard Assessment for the Main Seismogenic Zones in North Algeria*, Pure Appl. Geophys. *152*, 281–314.

HAMDACHE, M., BEZZEGHOUD, M., and MOKRANE, A. (1998), *Estimation of Seismic Hazard Parameters in the Northern Part of Algeria*, Pure Appl. Geophys. *151*, 101–117.

HAMILTON, R.M. (1967), *Mean Magnitude of an Earthquake Sequence*, Bull. Seismol. Soc. Am. *57*, 1115–1126.

JACKSON, D.D. and KAGAN, Y.Y. (1999), *Testable Earthquake Forecasts for 1999*, Seism. Res. Lett. *70*, 393–403.

JIN, A. and AKI, K. (1988), *Spatial and Temporal Correlation between Coda Q and Seismicity in China*, Bull. Seismol. Soc. Am. *78*, 741–769.

KAGAN, Y.Y. (1991), *Seismic Moment Distribution*, Geophys. J. Int. *106*, 123–134.

KAGAN, Y.Y. (1994), *Observational Evidence of Earthquakes as a Nonlinear Dynamical Process*, Physica D *77*, 160–192.

KAGAN, Y.Y. (1997), *Seismic Moment-frequency Relation for Shallow Earthquakes: Regional Comparison*, J. Geophys. Res. *102*, 2835–2852.

KAGAN, Y.Y. (2002a), *Seismic Moment Distribution Revisited: I. Statistical Results* Geophys. J. Int. *148*, 521–542.

KAGAN, Y.Y. (2002b), *Seismic Moment Distribution Revisited: II. Moment Conservation Principle*, Geophys. J. Int. *149*, 731–754.

KENDALL, M.G. and STUART, A., *The Advanced Theory of Statistics. Vol. 2, Inference and Relationship* (Griffin, London 1967).

KIJKO A. and GRAHAM, G. (1998), *"Parametric-Historic" Procedure for Probabilistic Seismic Hazard Analysis. Part I: Assessment of Maximum Regional Magnitude* m_{max}, Pure Appl. Geophys. *152*, 413–442.

KIJKO, A., and SELLEVOLL, M.A. (1989), *Estimation of Earthquake Hazard Parameters from Incomplete Data Files. Part I, Utilization of Extreme and Complete Catalogues with Different Threshold Magnitudes*, Bull. Seismol. Soc. Am. *79*, 645–654.

KIJKO, A. and SELLEVOLL, M.A. (1992), *Estimation of Earthquake Hazard Parameters from Incomplete Data Files. Part II, Incorporation of Magnitude Heterogeneity*, Bull. Seismol. Soc. Am. *82*, 120–134.

KIJKO, A., LASOCKI, S., and GRAHAM, G. (2001), *Nonparametric Seismic Hazard Analysis in Mines*, Pure Appl. Geophys. *158*, 1655–1675.

LASOCKI, S., and WEGLARCZYK, S., *Complex and variable magnitude distribution of microtremors induced by mining: An example from a gold mine, South Africa*. In Abstracts XXVI General Assemb. ESC (Tel Aviv 1998) p. 37.

LEGRAS, J., *Methodes et Techniques de L'analyse Numerique* (Dunod, Paris 1971).

MAIN, I.G, IRVING, D., MUSSON, R., and READING, A. (1999), *Constrains on the Frequency-magnitude Relation and Maximum Magnitudes in the UK from Observed Seismicity and Glacio-isostatic Recovery Rates*, Geophys. J. Int. *137*, 535–550.

MAIN, I.G. (1996), *Statistical Physics, Seismogenesis and Seismic Hazard*, Rev. Geophys. *34*, 433–462.

MAIN, I.G. and BURTON, P.W. (1984a), *Information Theory and the Earthquake Frequency-magnitude Distribution*, Bull. Seismol. Soc. Am. *74*, 1409–1426.

MAIN, I.G. and BURTON, P.W. (1984b), *Physical Links between Crustal Deformation, Seismic Moment, and Seismic Hazard for Regions of Varying Seismicity*, J. R. Astr. Soc. *79*, 469–488.

MARZA, V.I., KIJKO, A., and MANTYNIEMI, P. (1991), *Estimate of Earthquake Hazard in the Vrancea (Romania) Region*, Pure Appl. Geophys. *131*, 143–154.

MCGARR, A. *Some applications of seismic source mechanism studies to assessing underground hazard, In Rockburst and Seismicity in Mines* (eds Gay, N.C. and Wainwright, E.H.) (Symp. Ser. No. 6, 199-208. S. Afric. Inst. Min. Metal., Johannesburg, 1984).

MOLCHAN, G., KRONROD, T., and PANZA, G.F. (1997), *Multi-scale Seismicity Model for Seismic Risk*, Bull. Seismol. Soc. Am. *87*, 1220–1229.

MORTGAT, C.P. and SHAH, H.C. (1979), *A Bayesian Model for Seismic Hazard Mapping*, Bull. Seismol. Soc. Am. *69*, 1237–1251.

MOTAZEDIAN, D., MOINFAR, A.A., and BEHNAM, M. (1997), *Seismic Hazard Analysis Based on Virtual Earthquakes Distribution*, Mahab Ghods Consulting Engineering Co., p. 10.

NORDQUIST, J.M. (1945), *Theory of Largest Values Applied to Earthquake Magnitudes*, Trans. Am. Geophys. Union *26*, 29–31.

NUTTLI, O.W. (1981), *On the Problem of Maximum Magnitude of Earthquakes*, USGS Open Report, 13 p.

PAGE, R. (1968), *Aftershocks and Microaftershocks*, Bull. Seismol. Soc. Am. *58*, 1131–1168.

PAPADOPOULOS, G.A. and KIJKO, A. (1991), *Maximum Likelihood Estimation of Hazard Parameters in the Aegean Area from Mixed Data*, Tectonophysics *185*, 277–294.

PAPASTAMATIOU, D. (1980), *Incorporation of Crustal Deformation to Seismic Hazard Analysis*, Bull. Seismol. Soc. Am. *70*, 1321–1335.

PISARENKO, V.F. (1991), *Statistical Evaluation of Maximum Possible Magnitude*, Izvestiya, Earth Physics *27*, 757–763.

PISARENKO, V.F., LYUBUSHIN, A.A., LYSENKO, V.B., and GOLUBIEVA, T.V. (1996), *Statistical Estimation of Seismic Hazard Parameters: Maximum Possible Magnitude and Related Parameters*, Bull. Seismol. Soc. Am. *86*, 691–700.

PISARENKO, V.F.and, SORNETTE, D. (2001), *Characterization of the Frequency of Extreme Events by the Generalized Pareto Distribution*, Pure Appl. Geophys., in print. (preprint at http://arXiv.org/abs/cond-mat/001168)

RHOADES, D.A., VAN DISSEN, R.J., and DOWRICK, D.J. (1994), *On the Handling of Uncertainties in Estimating the Hazard of Rupture on a Fault Segment*, J. Geophys. Res. *99*, B7, 13,701—13,712.

RICE, J. (1975), *Statistical Methods of Use in Analyzing Sequences of Earthquakes*, Geophys. J. R. Astr. Soc. *42*, 671–683.

ROBSON, D.S., and WHITLOCK, J.H. (1964), *Estimation of a Truncation Point*, Biometrika *51*, 33–39.

SCHWARTZ, D.P. and COPPERSMITH, K.J. (1984), *Fault Behaviour and Characteristic Earthquakes: Examples from the Wasatch and San Andreas Fault Zones*, J. Geophys. Res. *89*, 5681–5698.

SHANKER, D. (1998), Personal communication.

SILVERMAN, B.W., *Density Estimation for Statistics and Data Analysis* (Chapman and Hall, London 1986).

SINGH, S.K., RODRIQUES, M., and ESTEVA, L. (1983), *Statistics of Small Earthquakes and Frequency of Large Earthquakes along the Mexico Subduction Zone*, Bull. Seismol. Soc. Am. *73*, 1779–1796.

SLEJKO, D. and KIJKO, A. (1991), *Seismic Hazard Assessment for the Main Seismogenic Zones in the Eastern Alps*, Tectonophysics *191*, 165–183.

SÓLNES, J., HALLDÓRSSON, B., and SIGBJÖRNSSON, R., *Assessment of seismic risk based on synthetic and upgraded earthquake catalogs of Iceland*. In *Proc. 9th Int. Seminar on Earthquake Prognostics* (San José, Costa Rica 1994).

SORNETTE, D. and SORNETTE, A. (1999), *General Theory of the Modified Gutenberg-Richter Law for Large Seismic Moments*, Bull. Seismol. Soc. Am. *89*, 1121–1130.

STAVRAKASIS, G.N. and TSELENTIS, G.A. (1987), *Bayesian Probabilistic Prediction of Strong Earthquakes in the Main Seismic Zones of Greece*, Boll. Geof. Teor. Appl. *29*, 51–63.

STEIN, R.S. and HANKS, T.C. (1998), *$M \geq 6$ Earthquakes in Southern California during the Twentieth Century: No Evidence for a Seismicity or Moment Deficit*, Bull. Seismol. Soc. Am. *88*, 635–652.

STOCK C. and SMITH, E.G.C. (2002), *Comparison of Seismicity Models Generated by Different Kernel Estimations*, Bull. Seismol. Soc. Am. *92*, 3, 913–922.

STONE, C.J. (1984), *An Asymptotically Optimal Window Selection Rule for Kernel Density Estimates*, Ann. Statist. *12*, 1285–1297

TORO, G.R., ABRAHAMSON, N.A., and SCHNEIDER, J.F. (1997), *Model of Strong Ground Motions from Earthquakes in Central and Eastern North America: Best Estimates and Uncertainties*, Seism. Res. Lett. *68*, 41–57.

VERE-JONES, D. (1992), *Statistical methods for the description and display of earthquake catalogs*. In *Statistics in the Environmental and Earth Sciences* (A.T. Walden and P. Guttorp, Eds. (Arnold Publishers, London, 1992) pp. 220–246.

WARD, S.N. (1997), *More on M_{max}*, Bull. Seismol. Soc. Am. *87*, 1199–1208.

WEICHERT, D.H. (1980), *Estimation of Earthquake Recurrence Parameters for Unequal Observation Periods for Different Magnitudes*, Bull. Seismol. Soc. Am. *70*, 1337–1346.

WEICHERT, D.H. and KIJKO, A. (1989), *Estimation of Earthquake Recurrence Parameters from Incomplete and Variably Complete Catalogue*, Seism. Res. Lett. *60*, 28.

WEIMER, S. and WYSS, M. (1997), *Mapping the Frequency-magnitude Distribution in Asperities: An Improved Technique to Calculate Recurrence Times?*, J. Geophys. Res. *102*, 15, 115–15, 128.

WEISS, L. and WOLFOWITZ, J. (1973), *Maximum Likelihood Estimation of a Translation Parameter of a Truncated Distribution*, Ann. Statis. *1*, 944–947.

WELLS, D.L. and COPPERSMITH, K.J. (1994), *New Empirical Relationships among Magnitude, Rupture Length, Rupture Width, Rupture Area, and Surface Displacement*, Bull. Seismol. Soc. Am. *84*, 974–1002.

WESNOUSKY, S.G., SCHOLZ, C.H., SHIMAZAKI, C.H., and MATSUDA, T. (1983), *Earthquake Frequency Distribution and the Mechanics of Faulting*, J. Geophys. Res. *88*, 9331–9340.

WGCEP (Working Group on Central California Earthquake Probabilities) (1995), *Seismic Hazard in Southern California: Probable Earthquakes, 1994 to 2024*, Bull. Seism. Soc. Am. *85*, 379–439.

WOO, G. (1996), *Kernel Estimation Methods for Seismic Hazard Area Source Modelling*, Bull. Seismol. Soc. Am. *86*, 353–362.

WOODROOFE, M. (1972), *Maximum Likelihood Estimation of a Translation Parameter of a Truncated Distribution*, Ann. Math. Statist. *43*, 113–122.

WOODROOFE, M. (1974), *Maximum likelihood estimation of a translation parameter of a truncated distribution II*, Ann. Statist. *2*, 474–488.

YURUI, H. and TIANZHONG, Z. (1997), *The K Method for Estimating Earthquake Activity Parameters and Effect of the Boundary Uncertainty of the Source Region: Discussion on the Seismic Zoning Method*, Earthq. Res. in China *11*, 299–305.

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